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Numerical Evaluation of a Cauchy Principal Value Integral that Arises in a Problem Involving the Generation of Instability Waves

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The details of the computation of a quantity that arises in the analysis of the coupling between an external disturbance and an instability wave on a free shear layer that eminates from a smooth surface are given. This quantity entails a Cauchy principal value integral whose integrand is itself given in terms of Cauchy principal value integrals.

1. INTRODUCTION

The purpose of this article is to present the details of the computation of the quantity

$$E = X_+(1)\left[\frac{1}{i\pi}⨍_0^\infty \frac{1}{(1+t)\,X_+(t)}\frac{dt}{t-1} - 1\right] - \frac{1}{2}, \tag{1.1}$$

where $⨍$ denotes Cauchy principal value and

$$X_+(t) = \left|\frac{t-1}{t}\right|^s \sqrt{\frac{t}{1+t}}\exp\left\{\frac{1}{2\pi i}\left[f(t)+g(t)\right]\right\}, \tag{1.2}$$

with

$$s \equiv -\frac{1}{2\pi i}\log t, \tag{1.3}$$

$$f(t) = ⨍_0^1 \frac{\log x}{x-t}\,dx, \tag{1.4}$$

and

$$g(t) = ⨍_0^\infty \log\left(\frac{x}{1+x}\right)\frac{dx}{x-t}. \tag{1.5}$$

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The value of E turns out to be

$$E = -1.30656296... + i0.541196100....$$

This quantity arises in [1, 2] in the analysis of the coupling between an external disturbance and an instability wave on a free shear layer that eminates from a smooth surface. The disturbance is taken to have harmonic time dependence and to be of small amplitude, and only spatially growing instability waves are considered.

Neglecting viscous effects in the flow, and treating the unsteady motion as a linear perturbation, one obtains an inhomogeneous linear boundary value problem, the inhomogeneity being the result of the imposed disturbance. This problem has an infinite number of solutions of the form $\psi = \psi_{\mathrm{p}} + C\psi_{\mathrm{h}}$, where ψ_{p} is a particular solution that does not involve the instability wave, ψ_{h} is an eigensolution of the homogeneous problem (obtained by deleting the imposed disturbance) that involves the instability wave, and C is an arbitrary constant. It is shown in [1, 2] that both ψ_{p} and ψ_{h} are singular along the separation line (of the unperturbed flow) and that their singularities are of exactly the same nature. Viscous effects are important only along the separation line and they act in such a way as to choose that one solution ψ that has no singularities along the separation line. For this solution the constant C turns out to be

$$C = \delta^4 \frac{2}{\pi\sqrt{3}} A \left(\frac{3}{\lambda}\right)^6 \frac{i}{E}. \tag{1.6}$$

Here δ is a parameter related to the mean flow velocity, frequency of the disturbance, and a characteristic length of the body, and depends on the nature of the separation. A is the amplitude of the pressure gradient along the airfoil that would exist at the separation line if there were no separation in the flow. Finally λ is a constant that is determined by the behavior of the undisturbed flow at its separation line, and depends on the shape of the body and on the viscosity relative to the dimensions of the body. For more details see [1, 2].

2. Computation of $f(t)$

For $t > 0$ let us make the change of variable of integration $x = ty$ in (1.4). We obtain

$$\begin{aligned} f(t) &= \fint_0^{1/t} \frac{\log(ty)}{y-1}\,dy \\ &= (\log t) \fint_0^{1/t} \frac{dy}{y-1} + \int_0^{1/t} \frac{\log y}{y-1}\,dy, \end{aligned} \tag{2.1}$$

the second integral on the right-hand side being an ordinary integral. Using the facts that

$$\int_a^b \frac{dy}{y-c} = \log\left|\frac{b-c}{a-c}\right| \qquad \text{for all } c \in (-\infty, \infty), \tag{2.2}$$

and

$$\int_0^1 \frac{\log y}{y-1}\, dy = \frac{\pi^2}{6}, \tag{2.3}$$

(2.1) becomes

$$f(t) = \log t \log\left|\frac{t-1}{t}\right| + \frac{\pi^2}{6} - D\left(\frac{1}{t}\right), \tag{2.4}$$

where $D(z) = -\int_z^1 [\log y/(y-1)]\, dy$ is the Dilogarithm as defined in [3, p. 1004, formula 27.7.1]. The Dilogarithm, defined in slightly different forms in various sources, is a well-studied function, see [4], and very efficient algorithms for its computation are available; see e.g., [5, p. 310; 6].

Using some of the properties of $D(z)$, $f(t)$ can be expressed in terms of the function $p(s) = \sum_{n=1}^{\infty} s^n/n^2$ that is analytic in the complex s-plane cut along the line $[1, \infty)$. Thus by [3, formula 27.7.2], we have

$$f(t) = \log t \left|\frac{t-1}{t}\right| + \frac{\pi^2}{6} - p\left(1 - \frac{1}{t}\right), \qquad \frac{1}{2} \leqslant t < \infty. \tag{2.5}$$

Using first [3, formula 27.7.5], then [3, formula 27.7.3], and finally [3, formula 27.7.2], in (2.5), we also have

$$f(t) = -\frac{1}{2}(\log t)^2 + \frac{\pi^2}{3} - p(t), \qquad 0 < t < 1. \tag{2.6}$$

Finally, by substituting $(x-t)^{-1} = -\sum_{n=1}^{\infty} x^{n-1} t^{-n}$, $0 \leqslant x \leqslant 1$, $t > 1$, in (1.4), and integrating term by term, we obtain

$$f(t) = p\left(\frac{1}{t}\right), \qquad t > 1. \tag{2.7}$$

We note that (2.5)–(2.7) will be used in the analysis of the integrand of (1.1).

3. Computation of $g(t)$

Making the change of variable of integration $y = x/(1+x)$ and defining $u = t/(1+t)$, (1.5) becomes

$$g(t) = (1-u) \int_0^1 \log y \frac{dy}{(1-y)(y-u)}. \tag{3.1}$$

Substituting the partial fraction decomposition

$$\frac{1}{(1-y)(y-u)} = \frac{1}{1-u}\left(\frac{1}{1-y} + \frac{1}{y-u}\right) \tag{3.2}$$

in (3.1), and invoking (2.3) and (1.4), (3.1) becomes

$$g(t) = -\frac{\pi^2}{6} + f\left(\frac{t}{1+t}\right); \tag{3.3}$$

thus the computation of $g(t)$ can be accomplished trivially once we know how to compute $f(t)$.

4. Computation of E

As can be seen from (1.1), the difficult part of the computation of E entails the numerical evaluation of the Cauchy principal value integral

$$H = \int_0^\infty \frac{1}{(1+t)\, X_+(t)} \frac{dt}{t-1}. \tag{4.1}$$

Here we have to make sure that the integrand is such that the principal value is well defined, we should also locate the singular points of the integrand on $[0, \infty)$ if there are any, and know their exact nature.

As is clear from (4.1), apart from the pole at $t = 1$, the other singularities, if there are any, are introduced by $X_+(t)$, and it is this function that we analyze below.

From (1.2) and (1.3), $X_+(t)$ can be expressed as

$$X_+(t) = \sqrt{\frac{t}{1+t}} \exp\left\{\frac{1}{2\pi i} \phi(t)\right\}, \tag{4.2}$$

where

$$\phi(t)=f(t)+g(t)-\log t\log\left|\frac{t-1}{t}\right|$$
$$=f(t)+f\left(\frac{t}{1+t}\right)-\frac{\pi^2}{6}-\log t\log\left|\frac{t-1}{t}\right|, \tag{4.3}$$

the second equality being a consequence of (3.3).

On the interval $[\frac{1}{2},\infty)$ $f(t)$ is given by (2.5). Substituting (2.5) in (4.3), we obtain

$$\phi(t)=f\left(\frac{t}{1+t}\right)-p\left(1-\frac{1}{t}\right). \tag{4.4}$$

Our first observation from (4.2) and (4.4) is that $X_+(t)$ is analytic on $[\frac{1}{2},\infty)$, since both $f(t/(1+t))$ and $p(1-1/t)$ are. Thus $X_+(t)$ is analytic also in every neighborhood of $t=1$, and $X_+(1)\neq 0$, so that the Cauchy principal value across $t=1$ in (4.1) is well defined. The value of $X_+(1)$ can be computed easily from (4.4) as follows: By (2.5)

$$f\left(\frac{1}{2}\right)=\frac{\pi^2}{6}-p(-1)=\frac{\pi^2}{6}+(1-2^{1-2})\,\zeta(2)=\frac{\pi^2}{4}, \tag{4.5}$$

where $\zeta(s)$ is the Riemann Zeta function, see [3, p. 807]. Also $p(0)=0$. Thus

$$X_+(1)=\frac{1}{\sqrt{2}}\exp\left(-i\frac{\pi}{8}\right). \tag{4.6}$$

By (2.6) $f(t)$ is analytic on $(0,1)$, thus $f(t/(1+t))$ is analytic there, too. Consequently $\phi(t)$ and hence $X_+(t)$ are analytic on $(0,1)$. Since $X_+(t)$ is also analytic on $[\frac{1}{2},\infty)$ from the previous paragraph, $X_+(t)$ is therefore analytic on $(0,\infty)$ by analytic continuation.

Finally, we have to analyze the behavior of $X_+(t)$ for $t\to 0+$ and $t\to +\infty$.

For $t\to 0+$ $f(t)$ is given by (2.6), and $f(t/(1+t))$ is given by (2.6) with t there replaced by $t/(1+t)$. Thus

$$\phi(t)=\log t\log\left(\frac{1+t}{1-t}\right)+\frac{\pi^2}{2}-\frac{1}{2}[\log(1+t)]^2-p(t)-p\left(\frac{t}{1+t}\right), \tag{4.7}$$

from which we deduce that $\phi(t)=\pi^2/2+o(1)$ as $t\to 0+$. This shows that $X_+(t)$ does not oscillate as $t\to 0+$, although it has a branch point at $t=0$.

For $t\to +\infty$, $f(t)$ is given by (2.7). Since $t/(1+t)\to 1-$ as $t\to +\infty$, $\frac{1}{2}<t/(1+t)<1$ for t sufficiently large. Thus $f(t/(1+t))$ is given by (2.5), with t there replaced by $t/(1+t)$. Therefore

$$\phi(t)=\log t\log\left(\frac{t+1}{t-1}\right)+p\left(\frac{1}{t}\right)-p\left(-\frac{1}{t}\right), \tag{4.8}$$

from which we deduce that $\phi(t) = o(1)$ as $t \to +\infty$. This shows that $X_+(t)$ does not oscillate as $t \to +\infty$, although it has a branch point at $t = \infty$.

In summary we have shown that $\phi(t)$ hence $X_+(t)$ are analytic in every closed interval in $(0, \infty)$ and that the Cauchy principal value is well defined.

We now write (4.1) as

$$H = \left(\int_0^{1/2} + \fint_{1/2}^{3/2} + \int_{3/2}^{\infty} \right) \frac{1}{(1+t)\, X_+(t)} \frac{dt}{t-1}. \tag{4.9}$$

Using an idea due to Longman [7], every principal value integral of the form

$$I = \fint_{-a}^{a} \frac{F(x)}{x}\, dx \tag{4.10}$$

can be expressed as an ordinary integral as

$$I = \int_{-a}^{a} \frac{F(x) - F(-x)}{2x}\, dx. \tag{4.11}$$

If we now apply a Gauss–Legendre quadrature formula with an *even* number of abscissas, see [8, 9], then I can be evaluated without having to compute $F'(0)$, because 0 is not an abscissa for such a quadrature formula. Since this formula is of the form

$$\int_{-a}^{a} h(x)\, dx \simeq \sum_{j=1}^{2n} A_j h(x_j), \tag{4.12}$$

with $-a < x_1 < x_2 < \cdots < x_{2n} < a$, and $x_j = -x_{2n-j+1}$, $A_j = A_{2n-j+1}$, $j = 1, ..., n$, the application of (4.12) to (4.11) results in

$$I \simeq \textstyle\sum_{j=1}^{2n} A_j \left.\frac{F(x)}{x}\right|_{x = x_j}. \tag{4.13}$$

If $F(x)$ is analytic over $[-a, a]$, then $[F(x) - F(-x)]/x$ is analytic over $[-a, a]$, hence the error in the Gauss–Legendre quadrature formula (4.13) tends to zero like $e^{-\alpha n}$, where $\alpha > 0$ depends on the location of the singularities of $[F(x) - F(-x)]/x$ nearest $[-a, a]$. The integral

$$H_2 = \fint_{1/2}^{3/2} \frac{1}{(1+t)\, X_+(t)} \frac{dt}{t-1} \tag{4.14}$$

was computed using a Gauss–Legendre quadrature formula with 12 abscissas. (Note that $t = 1$ is the midpoint of the interval $[\frac{1}{2}, \frac{3}{2}]$ and this interval can be mapped to $[-\frac{1}{2}, \frac{1}{2}]$ by a shift in the variable t.)

As for the integral

$$H_1 = \int_0^{1/2} \frac{1}{(1+t)\,X_+(t)}\,\frac{dt}{t-1} = \int_0^{1/2} G(t)\,dt \tag{4.15}$$

it is clear that the singularity of $X_+(t)$ at $t=0$ makes the accurate evaluation of H_1 very difficult. To cope with this difficulty we make the change of variable $t=e^{-\rho}$ thus mapping $t=0$ to $\rho=+\infty$. H_1 now becomes

$$H_1 = \int_{\log 2}^{\infty} G(e^{-\rho})\,e^{-\rho}\,d\rho, \tag{4.16}$$

the integrand being analytic for $\log 2 \leqslant \rho < \infty$, and $G(e^{-\rho})\,e^{-\rho} = O(e^{-\rho/2})$ as $\rho \to +\infty$. Thus for $c>0$

$$\left| \int_c^{\infty} G(e^{-\rho})\,e^{-\rho}\,d\rho \right| \leqslant \frac{4}{1-e^{-c}}\,e^{-c/2}. \tag{4.17}$$

If we choose c large enough and approximate H_1 as

$$H_1 \simeq \int_{\log 2}^{c} G(e^{-\rho})\,e^{-\rho}\,d\rho, \tag{4.18}$$

the error in this approximation will be bounded by $\varepsilon_1 = 4e^{-c/2}/(1-e^{-c})$.

Finaly, the integral

$$H_3 = \int_{3/2}^{\infty} \frac{1}{(1+t)\,X_+(t)}\,\frac{dt}{t-1} = \int_{3/2}^{\infty} G(t)\,dt \tag{4.19}$$

converges very slowly since $G(t) = O(t^{-2})$ as $t \to +\infty$. Using the transformation of variable $t=e^{\rho}$, (4.19) becomes

$$H_3 = \int_{\log(3/2)}^{\infty} G(e^{\rho})\,e^{\rho}\,d\rho. \tag{4.20}$$

The integrand is again analytic for $\log\frac{3}{2} \leqslant \rho < \infty$ and $G(e^{\rho})\,e^{\rho} = O(e^{-\rho})$ as $\rho \to +\infty$. Thus for $d>0$

$$\left| \int_d^{\infty} G(e^{\rho})\,e^{\rho}\,d\rho \right| \leqslant \frac{2}{1-e^{-d}}\,e^{-d}. \tag{4.21}$$

Choosing d large enough we approximate H_3 as

$$H_3 \simeq \int_{\log(3/2)}^{d} G(e^{\rho})\,e^{\rho}\,d\rho, \tag{4.22}$$

the error in the approximation being bounded by $\varepsilon_3 = 2e^{-d}/(1-e^{-d})$.

We now choose c and d such that $\varepsilon_1 \leqslant \varepsilon$ and $\varepsilon_3 \leqslant \varepsilon$, where ε is a given level of accuracy. Then we compute the integrals given in (4.18) and (4.22) with a maximum error $\varepsilon' \leqslant \varepsilon$. Our computation of these integrals was performed by using the adaptive quadrature routine DCADRE from the IMSL Library. Specifically, we chose $c = 80$, $d = 40$, so that $\varepsilon_1 \simeq 10^{-17}$ and $\varepsilon_3 \simeq 10^{-17}$, and $\varepsilon' = 10^{-12}$.

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